Aitken method on method of successive substitutions for solving a system of Volterra-Fredholm integral equations of the second kind

Talaat I. Hasan [1]
Department of Mathematics, College of Education, University of Salahaddin-Erbil, Iraq
E-mail: italhat@yahoo.com

Shaharuddin Salleh[2]
Department of Mathematics, University Technology Malaysia
Johor Bahru, Malaysia
E-mail: ss@utm.my

Nejmaddin A. Suleiman[3]
Department of Mathematics, College of Education, University of Salahaddin-Erbil, Iraq
gardy.muhamad@yahoo.com

***Abstract* -In this paper, we consider the linear system of Volterra-Fredholm integral equations of the second kind (SVFIE-2). We proposed method of successive substitutions (MSS) to solve SVFIE-2 and Aitken method on method of successive substitutions (AM on MSS) for solving the problem. In addition, a few theorems and two new algorithms are introduced. They are supported by numerical examples and simulations using Matlab. The results are very reasonably good when compared with the exact solution.**

***Key words*: method of successive substitutions, Aitken Method, system Volterra-Fredholm integral equations of the second kind.**

1. Introduction

The theory and application of integral equations is a significant subject within applied mathematics. Integral equations of various kinds arise in many fields of science and engineering. In addition, integral equations have been one of the principal instruments in various fields of science like applied mathematics physics, biology and engineering. In addition, the method plays an important role in applied mathematical modeling of many fields and various important problems. It has many advantages witnessed by the increasing frequency of the integral equations in the literature and in many areas because some problems have their mathematical representation appear directly. Nevertheless, the theoretical and numerical results concerning this subject are vary rate [1, 2, 3, 4, 5].

On the other hand, the solution of VFIE-2 has been a subject of considerable interest [6]. The solutions of integral equations have a major role in the fields of science and engineering. A physical event can be modeled by the differential equation, an integral equation or an integro-differential equation or a system of these [7]. Differential equations with transformed argument or differential equations of neutral type can be transformed to Volterra-Fredholm integral equations [8]. In addition, the Volterra-Fredholm integral equations appear from the parabolic boundary value problems [9]. It focuses on the numerical method of solution [10]. During the last 15 years, significant progress has been made in the numerical analysis for the linear and nonlinear versions of equations [6, 11]. Various types of analytic methods and numerical methods were used to solve integral equations [12].

At the same time, the numerical methods take a significant place for finding approximate solution of integral equations. A computational approach to solve system of integral equations is an essential work in scientific research [13, 14]. Lovitt [15] used the method of successive substitutions to solve the linear volterra integral equations of the second kind. Waz waz [16] used MSS for solving linear Fredholm integral equation of the second the kind. Rahmani [17] used MSS to find approximate solution of Fredholm integral equation of the second kind. However, no work on the method of successive substitutions have been reposted.

We extend the work further by considering a system of linear Volterra-Fredholm integral equations of the second kind with symmetric kernels and solve it by using MSS and AM on MSS.

2. Definitions and theorems

Definition 1 [7, 18]:- The integral equation

$$y(s) = f(s) + \lambda\int_a^s k(s,t)y(t)dt + \lambda^*\int_a^b g(s,t)y(t)dt, \qquad (1)$$

is called linear Volterra-Fredholm integral equation of the second kind where the functions $k(s,t)$ and $g(s,t)$ are called kernels of integral equation, such that $f(s)$, $k(s,t)$ and $g(s,t)$ are known functions on $R = \{(s,t), a < t < s < b\}$, such that , λ, λ^* are constant and $y(s)$ is unknown function.

3. System of Volterra-Fredholm integral equations of the second kind

The system of Volterra-Fredholm integral equations of the second kind is given as follows

$$y_i(s) = f_i(s) + \sum_{k=1}^{m} \lambda_{ik}\int_a^s k_{ik}(s,t)y_k(t)dt + \sum_{k=1}^{m}\lambda^*_{ik}\int_a^b k^*_{ik}(s,t)y_k(t)dt, \quad (2)$$

for $i = 1,2,3,...,p$. $f_i(s)$ is a continuous function on $[a,b]$. $k_{ik}(s,t)$ and $k^*_{ik}(s,t)$ denotes the given continuous functions on $R = \{(s,t), a < t < s < b\}$ while $y_i(s)$ is the unknown function to be determined [4,7].

4. Method of Successive Substitutions

This method introduces the solution of the integral equation in a series form through evaluation single integral and multiple integrals as well [16].

$$y(s) = f(s) + \lambda\int_a^b k(s,t)y(t)dt, \qquad (3)$$

In this method, we put $s = t$ and $t = t_1$ in the Fredholm integral equation, we get

$$y(t) = f(t) + \lambda\int_a^b k(t,t_1)y(t_1)dt_1, \qquad (4)$$

Putting $y(t)$ on the right hand side of Equation (4) form its obtained value in (3) yields

$$y(s) = f(s) + \lambda\int_a^b k(s,t)[f(t) + \lambda\int_a^b k(s,t_1)y(t_1)dt_1]dt$$

$$= f(s) + \lambda\int_a^b k(s,t)f(t)dt + \lambda^2\int_a^b k(s,t)\int_a^b k(t,t_1)y(t_1)dt_1dt, \quad (5)$$

Substituting $s = t_1$ and $t = t_2$ in Equation (5) we obtain

$$y(t_1) = f(t_1) + \lambda\int_a^b k(t_1,t_2)y(t_2)dt_2$$
$$+ \lambda^2\int_a^b k(t_1,t_2)\int_a^b k(t_1,t_2)y(t_2)dt_2dt_1, \qquad (6)$$

Substituting the value of $y(t_1)$ from in Equation (6) into the right hand side of (5) leads to

$$y(s) = f(s) + \lambda\int_a^b k(s,t)f(t)dt + \lambda^2\int_a^b\int_a^b k(s,t)k(t,t_1)f(t_1)dt_1dt +$$
$$\lambda^3\int_a^b\int_a^b\int_a^b k(s,t)k(t,t_1)k(t_1,t_2)y(t_2)dt_2dt_1dt, \qquad (7)$$

Accordingly, the general series form $y(s)$ can be written as

$$y(s) = f(s) + \lambda\int_a^b k(s,t)f(t)dt + \lambda^2\int_a^b\int_a^b k(s,t)k(t,t_1)f(t_1)dt_1dt +$$
$$\lambda^3\int_a^b\int_a^b\int_a^b k(s,t)k(t,t_1)k(t_1,t_2)f(t_2)dt_2dt_1dt$$
$$+\lambda^4\int_a^b\int_a^b\int_a^b\int_a^b k(s,t)k(t,t_1)k(t_1,t_2)k(t_2,t_3)f(t_3)dt_3dt_2dt_1dt + \ldots \quad (8)$$

Theorem 1 [15]. The series solution method given in Equation (8) then it converges uniformly in the interval $[a,b]$ if $\lambda M(b-a) < 1$ where $|k(s,t)| < M$.

Theorem 2 [9]. Let $f(s)$ be a continuous function on bounded region then $f(s)$ is bounded.

Theorem 3. Let $f(s)$ be a continuous function on $[a,b]$ then $\int_a^b f(t)dt$ is bounded.

Proof. By suppose we have $f(s)$ be a continuous is function on $[a,b]$, then by theorem 2, $f(s)$ is bounded function,
then there exist appositive real number u such that $|f(s)| \le u$, $\forall s \in [a,b]$.
Then $-u \le f(s) \le u$ by taking the integration over $[a,b]$.
Then we get $-u\int_a^b dt \le \int_a^b f(t)dt \le u\int_a^b dt$

$$-u(b-a) \le \int_a^b f(t)dt \le u(b-a),$$

then we get $\int_a^b f(t)dt$ is bounded.

5. Using MSS to solve Volterra-Fredholm integral equation of the Second

Suppose that the linear Volterra- Fredholm integral equation of the second kind is given in Equation (1).

In MSS putting $s = t$ and $t = t_1$ in the VFIE-2, produces

$$y(t) = f(t) + \lambda\int_a^t k(t,t_1)y(t_1)dt_1 + \lambda^*\int_a^b g(t,t_1)y(t_1)dt_1, \qquad (9)$$

In this method, we substitute successively $y(t)$ and its value as given by Equation (1). We get

$$y(s) = f(s) + \lambda\int_a^s k(s,t)[f(t) + \lambda\int_0^t k(t,t_1)y(t_1)dt_1 + \lambda^*\int_a^b g(t,t_1)y(t_1)dt_1]dt)$$
$$+\lambda^*\int_a^b g(s,t)[f(t) + \lambda\int_a^t k(t,t_1)y(t_1)dt_1 + \lambda^*\int_a^b g(t,t_1)y(t_1)dt_1]dt$$
$$y(s) = f(s) + \lambda\int_a^s k(s,t)f(t)dt + \lambda^2\int_a^s k(s,t)\int_0^t k(t,t_1)y(t_1)dt_1dt +$$
$$\lambda\lambda^*\int_a^s k(s,t)\int_a^b g(t,t_1)y(t_1)dt_1dt + \lambda\int_a^b g(s,t)f(t)dt$$
$$+\lambda^*\lambda\int_a^b g(s,t)\int_a^t k(t,t_1)y(t_1)dt_1dt + (\lambda^*)^2\int_a^b g(s,t)\int_a^b g(t,t_1)y(t_1)dt_1dt, (10)$$

substituting $s = t_1$ and $t = t_2$ in Equation (10) we obtain

$$y(t_1) = f(t_1) + \lambda\int_a^{t_1} k(t_1,t_2)f(t_2)dt_2 + \lambda^2\int_a^{t_1} k(t_1,t_2)\int_0^{t_2} k(t_2,t_1)y(t_1)dt_1dt_2$$
$$+\lambda\lambda^*\int_a^{t_1} kt_1,t_2)\int_a^b g(t_2,t_1)y(t_1)dt_1dt + \lambda\int_a^b g(t_1,t_2)f(t_2)dt_2$$
$$+\lambda^*\lambda\int_a^b g(t_1,t_1)\int_a^{t_2} k(t_2,t_1)y(t_1)dt_1dt_2 + (\lambda^*)^2\int_a^b g(t_1,t_2)\int_a^b g(t_2,t_1)y_1(t)dt_1dt_2, \quad (11)$$

substituting the value of $y(t_1)$ from Equation (11) into the right hand side of Equation (10) we get

$$y(s) = f(s) + \lambda\int_a^s k(s,t) f(t)dt$$

$$+\lambda^2\int_a^s k(s,t)\int_a^{t_1} k(t,t_1) f(t_1)dt_1dt + \lambda^3\int_a^s k(s,t)\int_a^t k(t,t_1)\int_a^{t_1} k(t_1,t_2) f(t_2)dt_2dt_1dt$$

$$+\lambda^4\int_a^s k(s,t)\int_a^t k(t,t_1)\int_0^{t_1} k(t_1,t_2)\int_a^{t_2} k(t_2,t_1)y(t_1)dt_1dt_2dt_1dt$$

$$+\lambda^3\lambda^*\int_a^s k(s,t)\int_a^t k(t,t_1)\int_0^{t_1} k(t_1,t_2)\int_a^b g(t_2,t_1)y(t_1)dt_1dt_2dt_1dt$$

$$+\lambda^*\lambda^3\int_a^s k(s,t)\int_a^b k(t,t_1)\int_0^b g(t_1,t_2) f(t_2)dt_2dt_1dt$$

$$+\lambda^2(\lambda^*)^2\int_a^s k(s,t)\int_a^t k(t,t_1)\int_0^b g(t_1,t_2)\int_a^b k(t_2,t_1)y(t_1)dt_1dt_2dt_1dt$$

$$+\lambda^3\lambda^*\int_a^s k(s,t)\int_a^b g(t,t_1)\int_0^{t_1} k(t_1,t_2)\int_a^b k(t_2,t_1)y(t_1)dt_1dt_2dt_1dt$$

$$+\lambda^2(\lambda^*)^2\int_a^s k(s,t)\int_a^b g(t,t_1)\int_0^{t_1} k(t_1,t_2)\int_a^{t_2} k(t_2,t_1)y(t_1)dt_1dt_2dt_1dt$$

$$+\lambda^*\lambda^2\int_a^s k(s,t)\int_a^b g(t,t_1)\int_0^b g(t_1,t_2) f(t_2)dt_2dt_1dt$$

$$+\lambda(\lambda^*)^2\int_a^s k(s,t)\int_a^b g(t,t_1)\int_0^b g(t_1,t_2)\int_a^{t_2} k(t_2,t_1)y(t_1)dt_1dt_2dt_1dt$$

$$+\lambda(\lambda^*)^3\int_a^s k(s,t)\int_a^b g(t,t_1)\int_0^{t_1} g(t_1,t_2)\int_a^{t_2} g(t_2,t_1)y(t_1)dt_1dt_2dt_1dt$$

$$+\lambda\lambda^*\int_a^b g(s,t) f(t)dt \qquad +\lambda\lambda^*\int_a^b g(s,t)\int_a^t k(t,t_1) f(t_1)dt_1dt$$

$$+\lambda^*\lambda^2\int_a^s g(s,t)\int_a^t k(t,t_1)\int_0^{t_1} g(t_1,t_2) f(t_2)dt_2dt_1dt$$

$$+\lambda^*\lambda^3\int_a^s g(s,t)\int_a^b k(t,t_1)\int_0^{t_1} k(t_1,t_2)\int_a^b k(t_2,t_1)y(t_1)dt_1dt_2dt_1dt$$

$$+\lambda^2(\lambda^*)^2\int_a^s g(s,t)\int_a^t k(t,t_1)\int_0^{t_1} k(t_1,t_2)\int_a^b g(t_2,t_1)y(t_1)dt_1dt_2dt_1dt$$

$$+\lambda^2(\lambda^*)^2\int_a^s g(s,t)\int_a^t k(t,t_1)\int_0^b g(t_1,t_2) f(t_2)dt_2dt_1dt$$

$$+\lambda(\lambda^*)^2\int_a^b k(s,t)\int_a^b g(t,t_1)\int_0^b g(t_1,t_2)\int_a^{t_2} k(t_2,t_1)y(t_1)dt_1dt_2dt_1dt$$

$$+\lambda(\lambda^*)^3\int_a^b k(s,t)\int_a^b g(t,t_1)\int_0^b g(t_1,t_2)\int_a^b g(t_2,t_1)y(t_1)dt_1dt_2dt_1dt$$

$$+(\lambda^*)^2\lambda^2\int_a^b g(t_1,t_2)\int_0^b g(t_2,t_1) f(t_1)dt_1dt$$

$$+\lambda^*\lambda^2\int_a^b g(s,t)\int_a^b g(t,t_1)\int_0^{t_1} k(t_1,t_2) f(t_2)dt_2dt_1dt$$

$$+\lambda^2(\lambda^*)^2\int_a^b g(t_1,t_2)\int_a^b g(t_2,t_1)\int_0^{t_1} k(t_1,t_2)\int_a^{t_2} k(t_2,t_1)y(t_1)dt_1dt_2dt_1dt$$

$$+\lambda(\lambda^*)^3\int_a^b g(t_1,t_2)\int_a^b g(t_2,t_1)\int_0^{t_1} k(t_1,t_2)\int_a^b g(t_2,t_1)y(t_1)dt_1dt_2dt_1dt$$

$$+\lambda(\lambda^*)^2\int_a^b g(s,t)\int_a^b g(t,t_1)\int_0^b g(t_1,t_2) f(t_2)dt_2dt_1dt$$

$$+\lambda(\lambda^*)^3\int_a^b g(t_1,t_2)\int_a^b g(t_2,t_1)\int_0^b g(t_1,t_2)\int_a^{t_2} k(t_2,t_1)y(t_1)dt_1dt_2dt_1dt$$

$$+(\lambda^*)^4\int_a^b g(t_1,t_2)\int_a^b g(t_2,t_1)\int_0^b g(t_1,t_2)\int_a^b g(t_2,t_1)y(t_1)dt_1dt_2dt_1dt \quad (12)$$

Continuing the process *n* times, we get

$$y(s) = \ldots + R_{n+1}(s) \quad ,$$

Where the reminder after *n* terms is

$$R_{n+1}(s) = \lambda^{n+1}\int_a^s\int_a^{t_1}\int_a^{t_2}\cdots\int_a^{t_{n-2}}\int_a^{t_{n-1}}\int_a^{t_n} k(t,t_1)k(t_1,t_2)k(t_2,t_3)..k(t_{n-2},t_{n-1})k(t_{n-1},t_n) f(t_n)dt_ndt_{n-1}dt_{n-2}...dt_2dt_1dt +$$

$$\lambda^n\lambda^*\int_a^s\int_a^{t_1}\int_a^{t_2}\cdots\int_a^{t_{n-2}}\int_a^{t_{n-1}}\int_a^{s} k(t,t_1)k(t_1,t_2)k(t_2,t_3)..k(t_{n-2},t_{n-1})k(t_{n-1},t_n) f(t_n)dt_ndt_{n-1}dt_{n-2}...dt_2dt_1dt +$$

$$\lambda^{n-1}(\lambda^*)^2\int_a^s\int_a^{t_1}\int_a^{t_2}\cdots\int_a^{t_{n-2}}\int_a^{s}\int_a^{s} k(t,t_1)k(t_1,t_2)k(t_2,t_3)..k(t_{n-2},t_{n-1})k(t_{n-1},t_n) f(t_n)dt_ndt_{n-1}dt_{n-2}...dt_2dt_1dt +$$

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$$\lambda^{\frac{n+1}{2}}(\lambda^*)^{\frac{n+1}{2}}\int_a^s\int_a^{b}\int_a^{b}\cdots\int_a^{b}\int_a^{b}\int_a^{b} k(t,t_1)g(t_1,t_2)k(t_2,t_3)..k(t_{n-2},t_{n-1})g(t_{n-1},t_n) f(t_n)dt_ndt_{n-1}dt_{n-2}...dt_2dt_1dt +$$

$$(\lambda^*)^{\frac{n+1}{2}}\lambda^{\frac{n+1}{2}}\int_a^b\int_a^{t_1}\int_a^{t_2}\cdots\int_a^{t_{n-2}}\int_a^{t_{n-1}}\int_a^{t_n} k(t,t_1)k(t_1,t_2)k(t_2,t_3)..k(t_{n-2},t_{n-1}) f(t_n)dt_ndt_{n-1}dt_{n-2}...dt_2dt_1dt +$$

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$$(\lambda^*)^n\lambda\int_a^s\int_a^{b}\int_a^{b}\cdots\int_a^{b}\int_a^{b}\int_a^{b} g(t,t_1)g(t_1,t_2)g(t_2,t_3)...g(t_{n-2},t_{n-1})g(t_{n-1},t_n) f(t_n)dt_ndt_{n-1}dt_{n-2}...dt_2dt_1dt +$$

$$(\lambda^*)^{n+1}\int_a^b\int_a^{b}\int_a^{b}\cdots\int_a^{b}\int_a^{b}\int_a^{b} g(t,t_1)g(t_1,t_2)g(t_2,t_3)...g(t_{n-2},t_{n-1})g(t_{n-1},t_n) f(t_n)dt_ndt_{n-1}dt_{n-2}...dt_2dt_1dt$$

(13)

Accordingly, the general series for $y(s)$ can be written as

$$y(s) = f(s) + \lambda\int_a^s k(s,t) f(t)dt + \lambda\lambda^*\int_a^s k(s,t)\int_0^t k(t,t_1) f(t_1)dt_1dt + \lambda\lambda^*\int_a^s k(s,t)\int_a^b g(t,t_1) f(t_1)dt_1dt)$$

$$+\lambda^*\int_a^b g(s,t) f(t)dt + \lambda^*\lambda\int_a^b g(s,t)\int_a^t k(t,t_1) f_1(t)dt_1dt + \lambda^*\int_a^b g(s,t)\int_a^b g(t,t_1) f(t_1)dt_1dt + \ldots +$$

$$(\lambda^*)^n\int_a^b\int_a^{b}\int_a^{b}\cdots\int_a^{b}\int_a^{b}\int_a^{b} g(t,t_1)g(t_1,t_2)g(t_2,t_3)...g(t_{n-2},t_{n-1}) f(t_{n-1})dt_{n-1}dt_{n-2}...dt_2dt_1dt$$

(14)

It follows that by evaluating the integral in (12) and adding the terms, we get the numerical solution of (1). It is to be noted that in this method the unknown function $y(s)$ is substituted by the known function $f(s)$ that simplifies the evaluation of the multiple integrals.

Here, the general term of the series (12) may be written as follows:

$$z_n(s)=\lambda^n\int_a^s\int_a^{t_1}\int_a^{t_2}\cdots\int_a^{t_{n-2}}\int_a^{t_{n-1}} k(t,t_1)k(t_1,t_2)k(t_2,t_3)..k(t_{n-3},t_{n-2})k(t_{n-2},t_{n-1})f(t_{n-1})dt_{n-1}dt_{n-2}...dt_2dt_1dt+$$

$$\lambda^{n-1}\lambda^*\int_a^s\int_a^{t_1}\int_a^{t_2}\cdots\int_a^{t_{n-2}}\int_a^{b} k(t,t_1)k(t_1,t_2)k(t_2,t_3)..k(t_{n-3},t_{n-2})k(t_{n-2},t_{n-1})f(t_{n-1})dt_{n-1}dt_{n-2}...dt_2dt_1dt+$$

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$$\lambda^{\frac{n}{2}}(\lambda^*)^{\frac{n}{2}}\int_a^s\int_a^{b}\int_a^{b}\cdots\int_a^{b}\int_a^{b} k(t,t_1)g(t_1,t_2)k(t_2,t_3)..k(t_{n-3},t_{n-2})g(t_{n-2},t_{n-1})f(t_{n-1})dt_{n-1}dt_{n-2}...dt_2dt_1dt+$$

$$(\lambda^*)^{\frac{n}{2}}\lambda^{\frac{n}{2}}\int_a^b\int_a^{t_1}\int_a^{t_2}\cdots\int_a^{t_{n-2}}\int_a^{t_{n-1}} g(t,t_1)k(t_1,t_2)g(t_2,t_3)..g(t_{n-3},t_{n-2})k(t_{n-2},t_{n-1})f(t_{n-1})dt_{n-1}dt_{n-2}...dt_2dt_1dt+$$

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$$(\lambda^*)^{n-1}\lambda\int_a^b\int_a^{b}\int_a^{b}\cdots\int_a^{b}\int_a^{s} g(t,t_1)g(t_1,t_2)g(t_2,t_3)..g(t_{n-2},t_{n-1})f(t_{n-1})dt_{n-1}dt_{n-2}...dt_2dt_1dt+$$

$$(\lambda^*)^{n}\int_a^b\int_a^{b}\int_a^{b}\cdots\int_a^{b}\int_a^{b} g(t,t_1)g(t_1,t_2)g(t_2,t_3)..g(t_{n-2},t_{n-1})f(t_{n-1})dt_{n-1}dt_{n-2}...dt_2dt_1dt$$

(15)

Since $k(s,t)$, $g(s,t)$ and $f(s)$ are continuous in R and B respectively, then by using theorem 3 we get $|k(s,t)|\le V_1\le 1$ and $|g(s,t)|\le V_2\le 1$ in R and $|f(s)|\le u$ in B, where $V_1,V_2,u\in R^+$, we obtain

$$|z_n(s)|\le u|\lambda^n|V_1^n\frac{s^n}{n!}+u|\lambda^{n-1}||\lambda^*|V_1^nV_2\frac{(b-a)s^{n-1}}{n!}+$$
$$\ldots+u|(\lambda^*)^{n-1}||\lambda|V_1\,V_2^{n-1}\frac{(b-a)^{n-1}s}{n!}+$$
$$u|(\lambda^*)^n|V_2^n\frac{(b-a)^n}{n!} \qquad (16)$$

The series, for which this last written positive constant is the general term, is convergent. Consequently, the series in Equation (12) is absolutely and uniformly convergent throughout. If Equation (2) has continuous solution, it must be expressed by Equation (11). If $y(s)$ is continuous in $[a,s]$ its absolute value has a maximum value of $D<1$. Then

$$|R_{n+1}(s)|\le u|\lambda^{n+1}|V_1^{n+1}\frac{D^{n+1}}{(n+1)!}+$$
$$u|\lambda^n||\lambda^*|V_1^nV_2\frac{(b-a)D^n}{(n+1)!}+\ldots+u|(\lambda^*)^n|V_1V_2^n\frac{(b-a)^nD}{(n+1)!}+$$
$$u|(\lambda^*)^{n+1}|V_2^{n+1}\frac{(b-a)^{n+1}}{(n+1)!} \qquad (17)$$

Also we have by theorem $\lim_{n\to\infty}\frac{1}{n!}=0,$ where n is a positive integer number. Hence $\lim_{n\to\infty}R_{n+1}(s)=0$

Hence, we obtain that the function $y(s)$ verifying Equation (10), is the continuous function given by the series (11). Consequently, the function $y(s)$ given by (10) is precisely the solution of Equation (1), and we have the following theorem:

Theorem 4. Let $k(s,t)$ and $g(s,t)$ be a kernels of VFIE-2 and continuous functions defined over region $R=\{(s,t),a<t<s<b\ \}$ such that $|k(s,t)|\le V_1\le 1$ and $|g(s,t)|\le V_2\le 1$, $f(s)$ be the continuous functions defined over B such that $|f(s)|\le u$ then the VFIE-2 has one and only continuous solution $y(s)$ and this solution is given by absolutely and uniformly convergent series in Equation (12).

7. Using MSS to solve system of Volterra-Fredholm integral equation of the Second.

Suppose that we have SVFI-2, which is given in equation (2). We reformulate and employ MSS for solving equation (2) as follows, put $s=t$ and $t=t_1$ in the

SVFI-2, to get

$$y_i(t)=f_i(t)+\sum_{k=1}^m\lambda_{ik}\int_a^s k_{ik}(t,t_1)y_k(t_1)dt+\sum_{k=1}^m\lambda_{ik}^*\int_a^b k_{ik}^*(t,t_1)y_k(t_1)dt_1, \qquad (18)$$

In this method, we substitute successively $y_i(t)$ its value as given by equation (2). We obtain

$$y_i(s)=f_i(s)+\sum_{k=1}^m\lambda_{ik}\int_a^s k_{ik}(s,t)[f_i(t)+\sum_{k=1}^m\lambda_{ik}\int_a^s k_{ik}(t,t_1)y_k(t_1)+\sum_{k=1}^m\lambda_{ik}^*\int_a^b k_{ik}^*(t,t_1)y_k(t_1)dt_1]dt$$

$$+\sum_{k=1}^m\lambda_{ik}\int_a^b k_{ik}^*(s,t)[f_i(t)+\sum_{k=1}^m\lambda_{ik}\int_a^s k_{ik}(t,t_1)y_k(t_1)+\sum_{k=1}^m\lambda_{ik}^*\int_a^b k_{ik}^*(t,t_1)y_k(t_1)dt_1]dt$$

$$y_i(s)=f_i(s)+\sum_{k=1}^m\lambda_{ik}\int_a^s k_{ik}(s,t)f_i(t)dt+$$
$$\sum_{k=1}^m\lambda_{ik}\int_a^s k_{ik}(s,t)\sum_{k=1}^m\lambda_{ik}\int_a^s k(t,t_1)y(t_1)dt_1dt+$$
$$\int_a^s\sum_{k=1}^m\lambda_{ik}\int_a^s k(s,t)\sum_{k=1}^m\lambda_{ik}^*\int_a^b k_{ik}^*(t,t_1)y_k(t_1)dt_1dt$$
$$+\sum_{k=1}^m\lambda^*{}_{ik}\int_a^b k_{ik}^*(s,t)f(t)dt+$$
$$\sum_{k=1}^m\lambda^*{}_{ik}\int_a^b k_{ik}^*(s,t)\sum_{k=1}^m\lambda_{ik}\int_a^s k_{ik}(t,t_1)y(t_1)dt_1dt+$$
$$+\int_a^s\sum_{k=1}^m\lambda^*{}_{ik}\int_a^b k_{ik}^*(s,t)\sum_{k=1}^m\lambda_{ik}^*\int_a^b k_{ik}^*(t,t_1)y_k(t_1)dt_1dt \qquad (19)$$

Continue on this way, we get the general form of MSS for SVFI-2.

Algorithm for MSS

Step 1: Given a, b, n, Tol.

Step 2: Putting $s=t$ and $t=t_1$ in the VFIE-2, we obtain Equation (18).

Step 3: Substitute $y(t)$, in the right side of Equation (2).

Step 4:

While e_n > Tol

Step 4.1: Substitute $y(s)$, n times in the right side of the Equation (1) to get equation (19) if $n>1$.

Step 4.2: Calculate absolute error by $e_n = |y(s) - y_n(s)|$.

end while

Step 5: The numerical solution approach to the exact solution for $n \to \infty$.

6. Numerical examples on SVFIE-2

The method of this study is useful for finding the numerical solutions SVFIE-2. The computations associated with the examples were performed using MATLAB version 12. The examples are solved by using MSS.

Test Example 1. Consider the following SVFIE-2

$$y_1(s) = 2\sin(s) - \int_0^s y_2(t)dt$$

$$y_2(s) = \cos(s) - 0.4597 + \int_0^1 y_1(t)dt,$$

with the exact solutions $y_1(s) = \sin(s)$ and $y_2(s) = \cos(s)$.

Solution. By applying MSS and its program, we obtain the approximate solutions of $y_1(s)$ and $y_2(s)$ as follows and shown in Table 1.

$$y_1^1(s) = 2\sin(s)$$
$$y_2^1(s) = \cos(s) - 0.4597$$
$$y_1^2(s) = \sin(s) + (0.4597)s$$
$$y_2^2(s) = \cos(s) - 0.2298$$
$$y_1^3(s) = \sin(s) - (0.2298)s$$
$$y_2^3(s) = \cos(s) - 0.1149$$
$$y_1^4(s) = \sin(s) - (0.1149)s$$
$$y_2^4(s) = \cos(s) - 0.0575$$

Table 1. MSS results compared to the exact solution.

s	n	Exact solution of $y_1(s) = \sin(s)$	Approximate Value of $y_1(s)$	Absolute error $e_n = \|y_1(s) - y_1^n(s)\|$
	2	0.19866933	0.29060888	9.1939×10^{-2}
0.2	4		0.22165421	5.7462×10^{-2}
	6		0.20441555	1.1492×10^{-2}
	8		0.20010588	1.4365×10^{-3}
	10		0.19902846	3.5913×10^{-4}
	12		0.19875911	8.9784×10^{-5}
	14		0.19869177	2.2446×10^{-5}
	16		0.19867492	5.6115×10^{-6}
	18		0.19867073	1.4028×10^{-6}
	20		0.19866968	3.5072×10^{-7}
			0.19866941	8.7680×10^{-8}

s	n	Exact solution of $y_1(s) = \cos(s)$	Approximate Value of $y_2(s)$	Absolute error $e_n = \|y_2(s) - y_2^n(s)\|$
	2	0.98006693	1.20991542	2.2984×10^{-1}
0.2	4		1.03752878	5.7446×10^{-2}
	6		0.99443213	1.4365×10^{-2}
	8		0.98365796	3.5913×10^{-3}
	10		0.98096442	8.9784×10^{-4}
	12		0.98029103	2.2446×10^{-4}
	14		0.98012269	5.6115×10^{-5}
	16		0.98008060	1.4028×10^{-5}
	18		0.98007008	3.5072×10^{-6}
	20		0.98006745	8.7680×10^{-7}
			0.98006689	2.1920×10^{-8}

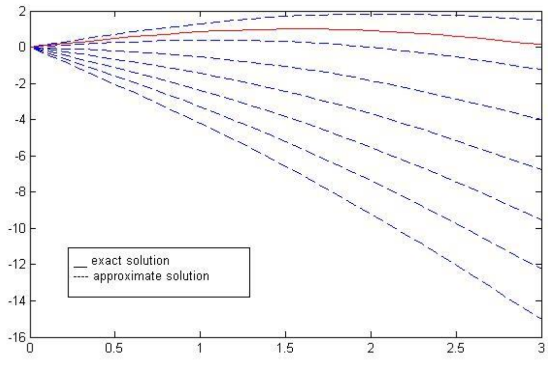

Fig. 1a. Graph $y_1(s) = \sin(s)$

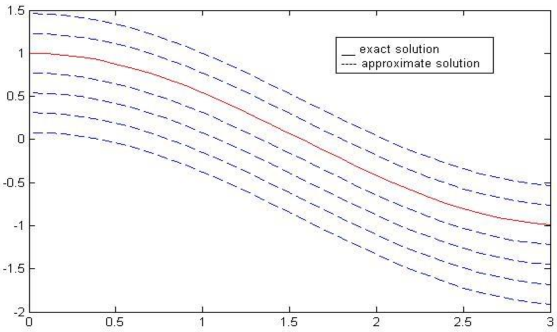

Fig. 1b. Graph $y_2(s)=\cos(s)$

Figures 1a and 1b show a comparison between the exact and the approximate solutions, given in Example 1.

Test Example 2. Consider the following SVFIE-2

$$y_1(s)=s+1+\frac{1}{2}\int_0^s y_1(t)dt-\int_0^1 (st)y_2(t)dt$$

$$y_2(s)=2+s+\int_0^s y_1(t)dt-\int_0^1 (st)y_2(t)dt$$

Where all kernels are Symmetric kernels with the exact solutions $y_1(s)=e^s$ and $y_2(s)=2e^s$.

Solution. By applying MSS and its program, we obtain the approximate solutions of $y_1(s)$ and $y_2(s)$ as follows and shown in Table 2.

$y_1^1(s)=1+s$

$y_2^1(s)=2+s$

$y_1^2(s)=1+(0.3333)s+(0.5000)s^2$

$y_2^2(s)=2+(0.3333)s+(0.5000)s^2$

$y_1^3(s)=1+(1.1528)s+(0.6667)s^2+(0.3333)s^3$

$y_2^3(s)=2+(2.0574)s+(0.3333)s^2+(0.1667)s^3$

$y_1^4(s)=1+(1.0574)s+(2.0574)s^2+(0.2222)s^3+(0.0833)s^4$

$y_2^4(s)=2+(20044)s+(1.0287)s^2+(0.1111)s^3+(0.0417)s^4$

Table 2. MSS results compared to the exact solution.

s	n	Exact solution of $y_1(s)=e^s$	Approximate Value of $y_1(s)$	Absolute error $e_n=\|y_1(s)-y_1^n(s)\|$
0.2	2	1.22140275	1.15333333	6.8069×10^{-2}
	4		1.23253333	1.1130×10^{-2}
	6		1.22198132	5.7856×10^{-4}
	8		1.22144734	4.4591×10^{-5}
	10		1.22140858	5.8275×10^{-6}
	12		1.22140355	7.9505×10^{-7}
	14		1.22140286	1.0760×10^{-7}
	16		1.22140277	1.4548×10^{-8}
	18		1.22140276	1.9672×10^{-9}
	20		1.22140275	2.6592×10^{-10}
			1.22140275	3.5961×10^{-11}

s	n	Exact solution of $y_1(s)=2e^s$	Approximate Value of $y_2(s)$	Absolute error $e_n=\|y_2(s)-y_2^n(s)\|$
0.2	2	2.44280551	2.15333333	2.3894×10^{-1}
	4		2.44297561	1.7010×10^{-4}
	6		2.44277599	2.9521×10^{-5}
	8		2.44279381	1.1697×10^{-5}
	10		2.44280390	1.6154×10^{-6}
	12		2.44280529	2.1672×10^{-7}
	14		2.44280548	2.9283×10^{-8}
	16		2.44280551	3.9598×10^{-9}
	18		2.44280551	5.354×10^{-10}
	20		2.44280551	7.239×10^{-11}
			2.44280551	9.790×10^{-12}

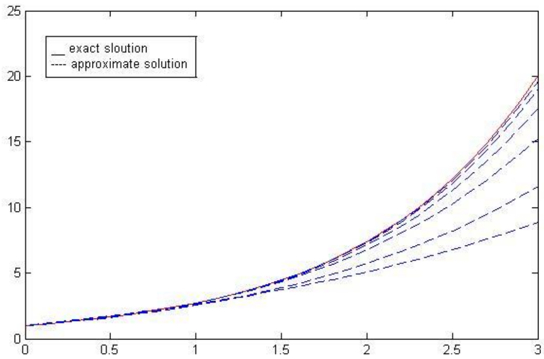

Fig. 2a. Graph $y_1(s)=e^s$

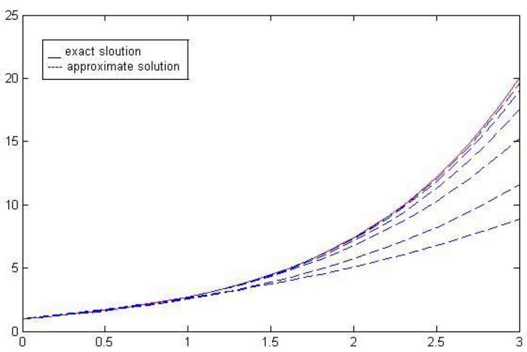

Fig. 2b. Graph $y_2(s)=2e^s$

Figures 2a and 2b show a comparison between the exact and the approximate solutions, given in Example 2.

8. New method: Aitken method on MSS for solving SVFIE-2.

In this section, we derive a new method, which has been used successfully for increasing the rate of convergence of MSS that is applied for solving SVFIE-2, which is Aitken method on MSS applied for solving SVFIE-2.

8.1 Aitken method

In this section, we have used a new technique to accelerate the iterative technique, called Aitkin method (AM) that can be used to accelerate the convergence of any sequence that is linearly convergence. Which we used it to solve VFIE-2. We have observed that the new technique is successful in the iterative process.

Let $\{y_i(s)\}_{i=0}^{\infty}$ be a linearly convergent sequence with limit p that is mean, for $e_i = y_{i+1} - p$

$\lim_{i\to\infty} \frac{|e_{i+1}|}{|e_i|} = l$ and $0 < l < 1.$

To investigate the construction of a sequence $\{y_i(s)\}_{i=0}^{\infty}$ the converges more rapidly to p let sufficiently large that the ratio can be used to approximate the limit. In addition, if we suppose that they have the same sign, then we get

$e_{i+1} = le_i$ and $e_{i+2} = le_{i+1}$

so $y_{i+2} \approx e_{i+2} + p \approx le_{i+1} + l$

or $y_{i+2} \approx l(y_{i+1} - p) + p$ (20)

Replacing $i+1$ by i in equation (20) we get

$$y_{i+1} \approx l(y_i - p) + p \quad (21)$$

For finding l, subtracting equation (21) from equation (20), gives

$$l = \frac{y_{i+2} - y_{i+1}}{y_{i+1} - y_i} \quad (22)$$

Substituting Equation (22) in Equation (20) yields

$$p = \frac{y_{i+2}y_i - y_{i+1}^2}{y_{i+2} - 2y_{i+1} + y_i}$$

$$= \frac{y_{i+2}^2 - y_{i+2}^2 + y_{i+2}y_i + 2y_{i+1}y_{i+1} - 2y_{i+1}y_{i+2} - y_{i+1}^2}{y_{i+2} - 2y_{i+1} + y_i}$$

$$= \frac{(y_{i+2}^2 + y_{i+2}y_i - 2y_{i+1}y_{i+1}) - (y_{i+2}^2 - 2y_{i+1}y_{i+2} + y_{i+1}^2)}{y_{i+2} - 2y_{i+1} + y_i}$$

$$= y_{i+2} - \frac{(y_{i+2}^2 + y_{i+2}y_i - 2y_{i+1}y_{i+2})}{y_{i+2} - 2y_{i+1} + y_i}$$

$$= y_{i+2} - \frac{(y_{i+2} - y_{i+1})^2}{y_{i+2} - 2y_{i+1} + y_i}$$

Hence

$$p = y_{i+2} - \frac{(y_{i+2} - y_{i+1})^2}{y_{i+2} - 2y_{i+1} + y_i}$$

$$= \frac{y_{i+2}y_i - y_{i+1}^2}{y_{i+2} - 2y_{i+1} + y_i}$$

Aitkin method is a based on the assumption that the sequence $\{y_i(s)\}_{i=0}^{\infty}$ defined by

$$\hat{y}_i = \frac{y_{i+2}y_i - y_{i+1}^2}{y_{i+2} - 2y_{i+1} + y_i}, \; i = 0,1,2,\ldots \quad (23)$$

This formula Converges faster than $\{y_i(s)\}_{i=0}^{\infty}$ to $p.$

Theorem 5. [19] Let $\{y_i(s)\}_{i=0}^{\infty}$ be a sequence of number that converges to a limit $p.$ then the new sequence

$$\hat{y}_i = \frac{y_{i+2}y_i - y_{i+1}^2}{y_{i+2} - 2y_{i+1} + y_i} \text{ for } n \geq 1$$

Converges *to* p faster than $\{y_i(s)\}_{i=0}^{\infty}$ if

$y_{i+1} - p = (\gamma - \delta_i)(y_i - p)$ where $|\gamma| < 1$ and

$\lim_{i\to\infty} \delta_i = 0$. *Implice that* $\frac{\hat{y}_i - p}{y_i - p} \to 0$ *as* $i \to \infty.$

Note :- let $\{y_i^n(s)\}_{n=0}^{\infty} = \{y_{i,n}(s)\}_{n=0}^{\infty}$

8.2 Aitkens on MSS applied for solving SVFIE-2.

Rewrite equation (23) as follows:

$$\hat{y}_{i,m} = \frac{y_{i,m+2}y_{i,m} - y_{i,m+1}^2}{y_{i,m+2} - 2y_{i,m+1} + y_{i,m}}, \quad (24)$$

find $y_{i,m}$, $y_{i,m+1}$ *and* $y_{i,m+2}$ for $i = 1,2,\ldots$ $, m = 0,1,2,\ldots$ by using the algorithm of MSS. put the results values in equation (24) to obtain new approximate solution $\hat{y}_{i,m}(s).$

Algorithm for AM on MSS:-

Step 1: Compute $y_{i,m}$, for $i = 1,2,3,\ldots p$ form Equation (19).

Step 2: Find $\hat{y}_{i,m}$ From Equation (24).

Step 3: Compute the value of absolute error by $e_m = \left|\hat{y}_{i,m}(s) - y_i(s)\right|$.

Step 4: The approximate solutions converges if $\lim_{m\to\infty}\left|y_i - \hat{y}_{i,m}\right| = 0$.

8.3 Numerical examples about SVFIE-2 and results by using AM on MSS.

The method of this study is very usefulness for finding the numerical solutions SVFIE-2. The computations associated with the examples were performed using MALAB version 12.

Test Example (3):- Find approximate solution of SVFIE-2 in example (1) by using AM on MSS

Solution: - Applying the numerical technique, which is, AM on MSS we obtained the results for approximate solutions of $y_1(s) = \sin(s)$ and $y_2(s) = \cos(s)$, as shown in table 3.

Table 3. AM on MSS results compared to the exact solution.

s	m	Exact solution of $y_1(s)=\sin(s)$	Approximate Values of $y_1(s)$	Absolute error $e_m=\lvert \hat{y}_{1,m}(s)-y_1(s)\rvert$
	2	0.19866933	0.20441555	5.7462×10^{-3}
0.2	4		0.20010588	1.4365×10^{-3}
	6		0.19902846	3.5913×10^{-4}
	8		0.19875911	8.9784×10^{-5}
	10		0.19869177	2.2446×10^{-5}
	12		0.19874923	5.6110×10^{-6}
	14		0.19866707	1.4028×10^{-6}
	16		0.19866968	3.5072×10^{-7}
	18		0.19866941	8.7680×10^{-8}
	20		0.19866933	1.753×10^{-11}

s	m	Exact solution of $y_1(s)=\cos(s)$	Approximate Values of $y_2(s)$	Absolute error $e_m=\lvert \hat{y}_{2,m}(s)-y_2(s)\rvert$
	2	0.98006657	0.93424234	1.4365×10^{-2}
0.2	4		0.98365792	3.5913×10^{-3}
	6		0.98096442	8.9784×10^{-4}
	8		0.98029103	1.7321×10^{-4}
	10		0.98012269	5.6115×10^{-5}
	12		0.98080606	1.4025×10^{-5}
	14		0.98007008	3.5072×10^{-6}
	16		0.98006745	8.7680×10^{-7}
	18		0.98006659	2.1920×10^{-9}
	20		0.98006657	5.480×10^{-11}

Test Example (4):- Find approximate solution of SVFIE-2 in example (2) by using AM on MSS.

Solution: - Applying the numerical technique, which is, AM on MSS we obtained the results for approximate solutions of $y_1(s)=\sin(s)$ and $y_2(s)=\cos(s)$, as shown in table 4.

Table 4. AM on MSS results compared to the exact solution.

s	m	Exact solution of $y_1(s)=e^s$	Approximate Value of $y_1(s)$	Absolute error $e_m=\lvert \hat{y}_{1,m}(s)-y_1(s)\rvert$
	2	1.22140275	2.22297036	1.5676×10^{-3}
	4		1.22142399	2.1235×10^{-5}
	6		1.22140272	7.7778×10^{-8}
	8		1.22140275	4.584×10^{-10}
	10		1.22140275	1.725×10^{-11}
	12		1.22140275	2.433×10^{-13}
	14		1.22140275	1.458×10^{-15}
	16		1.22140275	1.967×10^{-16}
	18		1.22140275	6.599×10^{-17}
	20		1.22140275	1.313×10^{-18}

s	m	Exact solution of $y_1(s)=2e^s$	Approximate Value of $y_2(s)$	Absolute error $e_m=\lvert \hat{y}_{2,m}(s)-y_2(s)\rvert$
	2	2.44280551	2.44690980	3.1142×10^{-3}
0.2	4		2.44281469	9.1799×10^{-6}
	6		2.44280553	1.8827×10^{-8}
	8		2.44280551	1.7542×10^{-9}
	10		2.44280551	7.172×10^{-11}
	12		2.44280551	4.134×10^{-13}
	14		2.44280551	4.557×10^{-15}
	16		2.44280551	5.334×10^{-16}
	18		2.44280551	7.239×10^{-17}
	20		2.44280551	9.790×10^{-18}

9. Conclusion

In this work, we propose two methods called MSS and AM on MSS for solving the problem. Several numerical examples were tested on applied algorithm of MSS and AM on MSS for solving SVFIE-2. From the results given in Tables 1, 2, 3 and 4, indicate clearly that both methods achieve good convergence as a number of iteration increases when the error decreases. The mentioned examples demonstrated the validity and applicability of the techniques. Finally, we concluded that AM on MSS converges faster than MSS as shown in the above tables for the number of iterations.

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